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Using Local Spectral Information in Domain Decomposition Methods – A Brief Overview in a Nutshell

Axel Klawonn[1,*], **Martin Kühn**[1,**], and **Oliver Rheinbach**[2,***]

[1] Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany

[2] Technische Universität Bergakademie Freiberg, Institut für Numerische Mathematik und Optimierung, Fakultät für Mathematik und Informatik, 09596 Freiberg, Germany

For second-order elliptic partial differential equations large discontinuities in the coefficients yield ill-conditioned stiffness matrices. The convergence of domain decomposition methods (DDM) can be improved by incorporating (numerically computed) local eigenvectors into the coarse space. Different adaptive coarse spaces for DDM have been constructed and used successfully. For two-level Schwarz, FETI-1 and BDD methods, adaptive coarse spaces with a rigorous theoretical basis are known for 2D and 3D. Although successfully in use for almost a decade, a theory for adaptive coarse spaces for FETI-DP and BDDC was lacking. While the problem was recently settled for 2D, the estimate for the 3D adaptive algorithm required improved coarse spaces. We give an brief overview of the literature, i.e., the different known approaches, and show numerical results for a specific adaptive FETI-DP method in 3D, where the condition number bound could only recently be proven.

Domain decomposition methods (DDM) are fast parallel iterative solution methods for the solution of implicit linear or linearized systems from the discretization of partial differential equations (PDEs). The convergence theory of these methods typically relies on a (global) condition number bound constructed from local theoretical estimates for finite element functions. Recently, new approaches have gained interest where these estimates are replaced by computable spectral bounds, e.g., from discrete local eigenvalue problems. A corresponding enrichment of the coarse problem then yields a global bound which only depends on a user-defined tolerance and some geometric bounds, e.g., the maximum number of edges or faces of a subdomain. Thus, convergence is not affected by ill-conditioning, e.g., from heterogeneities or (in some approaches) from almost incompressibility.

Already in [2], spectral information, i.e., the eigenvectors corresponding to the smallest eigenvalues of subdomain matrices, has been used with Neumann-Neumann methods, heuristically. In 2007, in [17], adaptive coarse spaces for FETI-DP and BDDC domain decomposition methods were proposed for 2D problems, at this time without a theoretical bound. Later, an adaptive coarse space for additive Schwarz methods was proposed [7, 8], based on eigenvalue problems on complete subdomains, replacing a Poincaré estimate. In [18], the strategy from [17] was extended to 3D and used very successfully, also in a parallel implementation. Later, coarse spaces based on local Dirichlet-to-Neumann maps were introduced for Schwarz preconditioners [6]. Then, for FETI-1 and BDD methods, a related adaptive approach was introduced [20]. At about the same time, published in [15], an adaptive approach for FETI-DP and BDDC methods was introduced by replacing a Poincaré inequality and an extension theorem (on edges) by eigenvalue problems; see [13] for the complete theory in 2D.

Only recently, in 2015, for the first time a rigorous condition number estimate was then proven in [14] for the widely used adaptive approach from [17], for 2D problems. Also in [14], a comparison of three adaptive coarse spaces for 2D problems, i.e., those of [17], [4, 12], and [13, 15], was provided, discussing their strengths and weaknesses, and considering their performance in numerical experiments. Then, but for an improved coarse space, in [11], a condition number estimate for 3D was shown for the first time; see also a remark in [5]. Moreover, strategies were proposed to reduce the number of eigenvalue problems. The improved coarse space was established by using an additional enrichment by certain eigenvectors on edges and is the first theory publically available for adaptive FETI-DP or BDDC methods in 3D.

For BDDC and FETI-DP, an adaptive coarse space for two dimensional problems has also recently been introduced in [10]. Recently, extensive new work on adaptive coarse spaces for BDDC in three dimensions has been presented; see the technical reports [1, 3, 19]. For Schwarz, see the recent report [9]. An overview of adaptive coarse spaces for BDDC was also given very recently, in a technical report [5].

At the center of the condition number estimates for FETI-DP and BDDC methods is the spectrum of the $P_D = B_D^T B$ operator; for more details, see [21]. Adaptive approaches can directly consider the P_D operator, using eigenvalue problems after localization. Alternatively, following standard theory, the spectrum of the P_D operator can be estimated by, e.g., replacing inequalities of Poincaré type as well as extension theorems by computable bounds. In this short paper, we consider our adaptive FETI-DP approach from [11] and apply it to linear elasticity on $\Omega = [0, 1]^3$ with zero Dirichlet boundary conditions on the face with $x = 0$. Then, let $f := [0.1, 0.1, 0.1]^T$, $\nu = 0.3$, and $E \in \{1, 1e6\}$. We could equally consider almost incompressible linear elasticity problems discretized by inf-sup stable finite elements or heterogeneous diffusion problems.

* Corresponding author: e-mail axel.klawonn@uni-koeln.de, phone +49 221 470 7868, fax +49 221 470 7871.

** Corresponding author: e-mail martin.kuehn@uni-koeln.de, phone +49 221 470 7860, fax +49 221 470 7871

*** Corresponding author: e-mail oliver.rheinbach@math.tu-freiberg.de, phone +49 3731 393279.

		FETI-DP w/ Vertex + Edge Average Coarse Space			FETI-DP w/ Extended Vertex + Adaptive Coarse Space [11]			
N	$\|\lambda\|$	$\|\Pi\|$	cond	iter	$\|\Pi\|$	$\|U\|$	cond	iter
4^3	20,991	2,367	1.10e+06	>2,000	2,517	1,761	9.74	36
6^3	80,199	9,168	1.57e+06	>2,000	10,110	5,514	10.04	36

Table 1: Linear elastic composite material with $N^{2/3}$ many stiff beams spanning from the face with $x = 0$ to the face with $x = 1$; discretized by $H/h = 6$ and $\mathcal{P}_1$ finite elements. Since the regular case can be solved already reliably by the classic algorithm of [18], we here use an (irregular) METIS decomposition and follow the new approach in [11] with TOL $= 10$.

For the FETI-DP method, we decompose Ω into N nonoverlapping subdomains Ω_i, $i = 1, \ldots, N$, where Γ is the interface. The FETI-DP method is based on the system $F\lambda = d$ with $F = B\widetilde{S}^{-1}B^T$ and an initial coarse space defined by an operator $\widetilde{S}_{\Pi\Pi}$ acting on the primal variables $\widetilde{u}_\Pi$. In our case, the operator $\widetilde{S}_{\Pi\Pi}$ is related to, at least, all vertex variables; cf. Table 1. We also demand that every curved edge in 3D has three primal vertex variables, to avoid rigid body (hinge) modes around the edges; see also [17]. Other approaches are also possible. By introducing an adequate scaling as well as extension and restriction operators R_Γ, R_Γ^T, we can also use the standard Dirichlet preconditioner $\widehat{M}^{-1} = B_D R_\Gamma^T S R_\Gamma B_D^T$. When estimating the Rayleigh quotient of the preconditioned system, i.e., $\langle F\lambda, \widehat{M}^{-1}F\lambda\rangle \leq C\langle F\lambda, \lambda\rangle$, we see that the $P_D = B_D^T B$ operator comes into play since the left hand side consists of an inner product defined by $P_D^T R_\Gamma^T S R_\Gamma P_D$; see, e.g., [11].

We consider generalized local eigenvalue problems on faces shared by subdomains (see [17,18] or, in our notation, in [14]) and also on edges, which is crucial to obtain the theoretical bound in 3D; see [11]. For a closed face or edge shared by two subdomains Ω_i and Ω_j, we then introduce localized versions of operators of the preconditioned FETI-DP system. The local jump operator B_{ij} consists of all the rows of B containing exactly one $+1$ and one -1. On the other hand, $B_{D_{ij}}$ will be the scaled version of B_{ij} restricting the global scaling to either the closed face or edge under consideration. The matrix $S_{ij} = \text{blockdiag}(S_i, S_j)$ is a blockdiagonal matrix of the uncoupled local Schur complements on the interface Γ, and we define the bilinear form $s_{ij}(\cdot,\cdot) := (\cdot, S_{ij}\cdot)$ for $v_{ij} \times w_{ij}$ with $v_{ij}, w_{ij} \in W_i \times W_j$. The local generalized eigenvalue problems can then be expressed by the variational formulation: Find $w_{ij}^k \in (\ker S_{ij})^\perp$, such that

$$s_{ij}(B_{D_{ij}}^T B_{ij} v_{ij}, B_{D_{ij}}^T B_{ij} w_{ij}^k) = \mu_{ij}^k s_{ij}(v_{ij}, w_{ij}^k) \quad \forall v_{ij} \in (\ker S_{ij})^\perp, \tag{1}$$

and we use the eigenvectors w_{ij}^k for $\mu_{ij}^k \geq$ TOL in order to compute the coarse space elements (constraints) $u_{ij}^k := B_{D_{ij}} S_{ij} B_{D_{ij}}^T B_{ij} w_{ij}^k$. This approach is different from [13], where two types of eigenvalue problems were used. Then, coarse elements defined on the closure of a face will be split such that all the constraints are defined on disjunct geometrical entities, i.e., either on open faces or edges. Let us note that there are strategies available to reduce the size of the coarse space presented here while keeping a reliable algorithm. These strategies may discard certain eigenvalue problems and/or constraints; cf. [11].

The adaptively computed constraints are enforced by a deflation approach, in our case, the balancing preconditioner. Other approaches are also possible. For balancing and deflation in the context of FETI-DP and BDDC, see [16]. Our numerical results show that the condition number can successfully be controlled for our heterogeneous problems in 3D, in accordance with the theory [11].

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